

D-brane Solutions in Non-Commutative Gauge Theory on Fuzzy Sphere

Koji HASHIMOTO^{*a} and Kirill KRASNOV^{†b}

^a*Institute for Theoretical Physics,
University of California, Santa Barbara, CA 93106*

^b*Department of Physics,
University of California, Santa Barbara, CA 93106*

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Abstract

Non-commutative gauge theory on fuzzy sphere was obtained by Alekseev *et al.* as describing the low energy dynamics of a spherical D2-brane in S^3 with the background b -field. We identify a subset of solutions of this theory which are analogs of “unstable” solitons on a non-commutative flat D2-brane found by Gopakumar *et al.* Analogously to the flat case, these solutions have the interpretation as describing D0-branes “not yet dissolved” by the D2-brane. We confirm this interpretation by showing the precise agreement of the binding energy computed in the non-commutative and ordinary Born-Infeld descriptions. We then study stability of the solution describing a single D0-brane off a D2-brane. Similarly to the flat case, we find an instability when the D0-brane is located close to the D2-brane. We furthermore obtain the complete mass spectrum of 0-2 fluctuations, which thus gives a prediction for the low energy spectrum of the 0-2 CFT in S^3 . We also discuss in detail how the instability to a formation of the fuzzy sphere modifies the usual Higgs mechanism for small separation between the branes.

^{*}koji@itp.ucsb.edu

[†]krasnov@cosmic.physics.ucsb.edu

1 Introduction

In recent years, much of the progress of string theory was based on a study of statics and dynamics of D-branes in various situations. A particularly interesting recent development is the introduction of non-commutativity into the D-brane worldvolume theory [1, 2, 3]. The non-commutativity makes transparent the relation between D-branes of different worldvolume dimensions through the construction of various solitons [4, 5, 6, 7, 8], which can be thought of as a manifestation of the “brane democracy” [9].

So far, most of the attention was given to the case of constant non-commutativity, which is what one has for the flat D-branes in a non-varying NS-NS 2-form field. Potentially interesting is the more general case of a non-vanishing field strength. Such backgrounds are necessarily curved, which makes the analysis more difficult. The potential payoff here is that one may be able to observe aspects of brane dynamics which are absent in the flat case.

One of the simplest backgrounds of this type is $S^3 \times M_7$, where M_7 is some 7-dimensional manifold. This background can be realized in string theory as the near-horizon geometry of a stack of NS5-branes. There exists an exact description of D2-branes on S^3 in terms of SU(2) WZW model. One finds, see [10, 11, 12, 13], that supersymmetric D2-branes wrap certain conjugacy classes in SU(2). These are certain integral spheres in S^3 . For radii of the spherical D2-brane much smaller than the radius of curvature of S^3 , and in the Seiberg-Witten (SW) limit $\alpha' \rightarrow 0$, the geometry of D2-brane worldvolume becomes [11] that of a fuzzy sphere [14]. As was shown in [15], the low-energy dynamics in this case is described by a certain non-commutative gauge theory on fuzzy S^2 . The action of this theory contains, apart from the usual Yang-Mills (YM) term, the Chern-Simons (CS) term. This is one of the examples in which YM-CS theory appears in string theory context [16, 17].

A large set of solutions of this theory was obtained in [15]. The solutions were interpreted as describing stacks of D2-branes of various radii. As we show in this paper, there is a particular interesting and simple subset of solutions of [15] that describe a single D2-brane together with a number of D0-branes. These solutions are exact analogs (in 2+1 dimensions) of the non-commutative monopole solution found in [5] and of the “unstable” solitons that were found in [7] for the case of a flat D2-brane. We describe this set of solutions, and confirm their D0-brane interpretation by comparing the energy obtained in the non-commutative gauge theory to that found in the ordinary Born-Infeld (BI) description.

The non-commutative solution describing co-centric D2-brane shells was argued to be stable in [15]. In the present paper we extend the analysis of stability to the case of non-cocentric shells, considering the subset of solutions that have the D0-brane interpretation. We find that, when a D0-brane is located too close to the shell of the D2-brane, the system

becomes unstable to forming a larger D2-brane. Furthermore, we completely diagonalize the set of 0-2 fluctuation modes. We find a discrete, finite set of massive modes. This gives a prediction for the spectrum of light modes of the 0-2 worldsheet CFT on S^3 in the SW limit.

Our results on stability of the configuration consisting of a single D2-brane together with a D0-brane have interesting applications for the phenomenon of gauge symmetry breaking. Consider Dp -branes put on $S^3 \times M_7$ in such a way that the directions along the branes are all in M_7 . In the usual flat case D-brane Higgs mechanism the gauge symmetry on the Dp -brane worldvolume is broken by separating the branes, and the masses of gauge bosons are proportional to the distance between them [18]. In our case, the usual mechanism is valid for large enough separation of the branes in S^3 . Thus, for two Dp -branes that are separated in S^3 by a large enough distance the worldvolume gauge symmetry is broken from $U(2)$ to $U(1)$, as usual. However, when the branes are located too close, this usual mechanism is no longer valid. Indeed, as our analysis shows, two D0-branes in S^3 located too close will form a fuzzy sphere (of smallest non-trivial radius). This breaks the $U(2)$ symmetry completely. Thus, as the separation between the branes decreases one gets a complete symmetry breaking, instead of the expected restoration of symmetry, as in the flat case. Therefore the phenomenon of polarization of branes [19, 20, 21] leads to a complete breaking of the worldvolume gauge symmetry and in particular prevents the symmetry restoration.

The organization of the paper is as follows. In the next section, for readers convenience, we review some known material. We first give, following [12] a brief account of the ordinary BI (Born-Infeld) description of spherical D2-brane in S^3 in the background b -field. The second part of this section reviews some facts about the non-commutative gauge theory of [15]. Here we give the action and describe what is known about the solutions. In section 3 we fix the coefficient in front of the action of [15], which is left unspecified in that paper. We shall need the precise normalization of the action to compare the energy calculated in the commutative and non-commutative descriptions. Section 4 describes in detail the set of solutions in question and gives their interpretation in terms of D0-branes. In section 5 we study fluctuations about these solutions, analyze stability and obtain the mass spectrum of fluctuations. In section 6 we discuss our results in view of the phenomenon of symmetry breaking. We conclude with a discussion.

While writing this paper we received the paper [22], which also considers the non-commutative gauge theory on fuzzy sphere, and studies fluctuations around certain solutions (one describing a single D2-brane and one describing two D0-branes). Since no comments are made in that paper as to the stability of the later system, the overlap with the present paper is rather marginal.

2 Review: Ordinary BI Description, Non-Commutative Gauge Theory on Fuzzy Sphere and Solutions

A spherical D-brane tends to shrink to minimize its energy. However, in the presence of some background fields that couple to the brane, a stabilization mechanism is known to work [19, 20, 12, 13, 21]. The simplest case is that of a D2-brane in the background of either a non-trivial R-R 3-form field, or NS-NS 2-form field with a non-vanishing field strength. The later situation is realized in the background $S^3 \times M_7$, which appears as the near-horizon geometry of a stack of NS5-branes. Since there exists the NS-NS b -field flux on S^3 , the D2-branes on it are possibly stabilized with a definite extent. D2-branes on S^3 were studied in the ordinary BI description in [12, 13], and using a non-commutative worldvolume theory in [11, 15]. Here we review some of the facts that will be needed in the following.

2.1 Ordinary BI description

In this subsection, we shall review Ref. [12] briefly, which treats D2-branes in S^3 using BI theory (see also [13]). The metric on S^3 is given by

$$ds^2 = k\alpha' (d\psi^2 + \sin^2 \psi (d\theta^2 + \sin^2 \theta d\varphi^2)). \quad (2.1)$$

Here k is the number of NS5-branes that were used to obtain the background. Then $k\alpha'$ is the radius of S^3 . The NS-NS 2-form field strength is proportional to the volume form of S^3 and is given by

$$H = 2k\alpha' \sin^2 \psi \sin \theta d\psi \wedge d\theta \wedge d\varphi. \quad (2.2)$$

The corresponding NS-NS 2-form potential $b : db = H$ is given by

$$2\pi\alpha' b = k\alpha' \left(\psi - \frac{\sin 2\psi}{2} \right) \sin \theta d\theta \wedge d\varphi. \quad (2.3)$$

The D2-branes in this background are stabilized by the flux of the U(1) gauge field. This flux can take integral values corresponding to different U(1) bundles one can have over S^2 . Thus,

$$F = -\frac{n}{2} \sin \theta d\theta \wedge d\varphi, \quad (2.4)$$

so that

$$\int_{S^2} F = -2\pi n. \quad (2.5)$$

The energy of a spherical D2-brane with n units of flux on it located at $\psi = \text{const}$ is then given by the BI action. Substituting (2.1), (2.3) and (2.4) into the D2-brane action, one gets

for the energy

$$E_n(\psi) = k\alpha'T_{(2)} \int d\theta d\varphi \sin\theta \sqrt{\sin^2\psi + \left(\psi - \frac{\sin 2\psi}{2} - \frac{\pi n}{k}\right)^2}. \quad (2.6)$$

Here $T_{(2)}$ is the D2-brane tension. The energy is minimized by

$$\psi_n = \frac{\pi n}{k}. \quad (2.7)$$

The corresponding energy is

$$E_n \equiv E_n(\psi_n) = 4\pi k\alpha'T_{(2)} \sin \frac{\pi n}{k}. \quad (2.8)$$

Thus, the stable configuration, corresponding to the minimum of the energy, is a spherical D2-brane, whose radius depends on the amount of the U(1) flux on it. We will use this formula for the energy in section 4.

2.2 Non-commutative description

Here we review Ref. [15] in the amount we need for the following. According to this paper, the low energy effective theory on a stack of N D2-branes of size n is given, in the $n/k \ll 1$ limit, by a non-commutative U(N) gauge theory on a fuzzy sphere of “radius” n . The action, as proposed by [15], is given by the sum of the YM and the CS terms. The YM term reads

$$S_{\text{YM}} = \frac{1}{4}(2\pi\alpha')^2 \frac{1}{\text{Dim}} \text{Tr} \mathbf{F}_{ab} \mathbf{F}^{ab}. \quad (2.9)$$

Here one raises and lowers indices using the metric $\mathbf{g}^{ab} = (2/k)\delta^{ab}$. We have restored the factor of $(2\pi\alpha')^2$ put to unity in [15], and explicitly specified that one uses the normalized trace. The curvature \mathbf{F}^{ab} of a non-commutative U(N) connection \mathbf{A}^a is most easily expressed in terms of the “covariant coordinates” [25] \mathbf{B}^a

$$\mathbf{B}^a = \mathbf{Y}^a + \mathbf{A}^a. \quad (2.10)$$

Here $\mathbf{Y}^a = Y^a/\sqrt{2\alpha'}$, and Y^a are the fuzzy sphere coordinates satisfying the usual SU(2) commutation relations

$$[Y^a, Y^b] = i\epsilon^{abc}Y^c. \quad (2.11)$$

Here ϵ^{abc} is the Levi-Civita symbol, and the contraction over the repeated index is c assumed. Using the covariant coordinate \mathbf{B}^a , the field strength can be written as

$$\mathbf{F}^{ab} = i[\mathbf{B}^a, \mathbf{B}^b] + \mathbf{f}^{ab}{}_c \mathbf{B}^c, \quad (2.12)$$

where $\mathbf{f}^{ab}_c = (2/k)\epsilon^{ab}_c/\sqrt{2\alpha'}$. The Chern-Simons term is best written in terms of the covariant coordinate \mathbf{B}^a and reads

$$S_{\text{CS}} = -\frac{i}{2}(2\pi\alpha')^2 \frac{1}{\text{Dim}} \text{Tr} \left(\frac{1}{3} \mathbf{f}^{abc} \mathbf{B}_a \mathbf{B}_b \mathbf{B}_c - \frac{i}{\alpha' k} \mathbf{B}_a \mathbf{B}^a + \frac{i}{3\alpha' k} \mathbf{Y}_a \mathbf{Y}^a \right). \quad (2.13)$$

The full action, written in terms of the covariant coordinate \mathbf{B}^a , then takes the form

$$(2\pi\alpha')^2 \frac{1}{\text{Dim}} \text{Tr} \left(-\frac{1}{4} [\mathbf{B}_a, \mathbf{B}_b] [\mathbf{B}^a, \mathbf{B}^b] + \frac{i}{3} \mathbf{f}^{abc} \mathbf{B}_a [\mathbf{B}_b, \mathbf{B}_c] + \frac{1}{6\alpha' k} \mathbf{Y}_a \mathbf{Y}^a \right). \quad (2.14)$$

Note that the last term is exactly such that the action takes zero value when \mathbf{B}^a is an irreducible representation $\mathbf{B}^a = \mathbf{Y}^a$, that is on the vanishing connection.

Having described the action for the non-commutative gauge theory in question, let us review what is known about its solutions. The equations of motion that one derives from (2.14) read

$$[\mathbf{B}^a, \mathbf{F}_{ab}] = 0. \quad (2.15)$$

If one uses this gauge theory to describe a stack of N D2-branes wrapping a fuzzy sphere of radius n , the “fields” entering the action and equations of motion are just matrices in $\text{Mat}_N \otimes \text{Mat}_n$, and $Y^a = \mathbf{1}_N \otimes Y_n^a$, where Y_n^a are generators in the n -dimensional irreducible representation. Ref. [15] describes a large set of solutions to (2.15). These are configurations for which the gauge field \mathbf{A} is constant $\mathbf{A}_a = \mathbf{S}_a$, that is, commutes with the fuzzy sphere coordinates: $[\mathbf{Y}_a, \mathbf{S}_b] = 0$. There are two types of such solutions: (i) \mathbf{S}_a is any set of commuting matrices from $\text{Mat}_N \otimes \mathbf{1}_n$. These matrices can then be diagonalized; eigenvalues have the obvious meaning of the coordinates of the centers of N fuzzy spheres in S^3 . (ii) $\mathbf{S}_a \in \text{Mat}_N \otimes \mathbf{1}_n$ is any, not necessarily irreducible, representation of $\text{SU}(2)$, so that it satisfies the usual commutation relation (2.11). Note that this type of solutions corresponds to flat connections $\mathbf{F}_{ab} = 0$. According to [15], such a solution corresponds to a fixed point of an RG flow that takes one from the original configuration of N branes of radius n centered around the origin to a new configuration consisting of branes of various radii, also centered around the origin. These branes can all have different radii, or some of them can be stacked, or the new configuration can be just a single brane. For the precise recipe for determining the final configuration see [15], formula (5.4). We shall give a somewhat different, although equivalent, description of a subset of the above solutions in section 4. As a final remark of this section, let us note that solutions for which \mathbf{B}^a is a representation (not necessarily irreducible), that is satisfy the commutation relation (2.11) are argued to be stable in [15]. These solutions describe flat connections. We shall analyze the stability of certain interesting non-flat solutions in section 5.

3 Normalization of the Action

Before we describe a subset of solutions that has a simple interpretation in terms of D0-branes, let us discuss the normalization of the action (2.14). It is clear that some factor of the brane tension must be included in the front. To fix it, we shall compare the YM part (2.9) of the action with the Yang-Mills action that can be obtained from the non-commutative BI action.

Let us, however, first rescale the fields for future convenience. We introduce new gauge field $A = \sqrt{2\alpha'} \mathbf{A}$ and the corresponding covariant coordinate field B and curvature F . Let us also lower the indices of one of the F 's in the action using the \mathbf{g} metric. We get

$$S_{\text{YM}} = \frac{(2\pi\alpha')^2}{(k\alpha')^2} \frac{1}{4} \frac{1}{\text{Dim}} \text{Tr} F_{ab} F_{ab}, \quad (3.1)$$

with the contraction over repeated indices assumed.

To fix the prefactor in front of this action, which, as we shall see, is proportional to a certain multiple of the $T_{(2)}$ (or $T_{(0)}$) brane tension, we would like to compare the above action with the one that can be obtained from the non-commutative BI action. The standard calculation gives

$$S_{\text{YM}} = \hat{T}_{(2)} (2\pi\alpha')^2 \int d^2x \sqrt{P(G)} \frac{1}{4} \text{Tr} F_{ab} F^{ab}. \quad (3.2)$$

Here $P(G)$ is the determinant of the pullback of the open string metric G_{ab} on S^2 . We have also introduced the “non-commutative” D2-brane tension

$$\hat{T}_{(2)} = \frac{1}{G_s (2\pi)^2 (\alpha')^{3/2}}, \quad (3.3)$$

where G_s is the open string coupling constant. Let us relate it to the closed string coupling constant g_s . A simple comparison of the ordinary and non-commutative BI actions gives [3]

$$G_s = g_s \left(\frac{\det P(G)}{\det P(g + 2\pi\alpha'(b + F))} \right)^{1/2}, \quad (3.4)$$

where P is the projection onto S^2 , and G is the open string metric. To calculate the right hand side we need to know G . This can be done using the formula for G in terms of $g, b + F$. As is shown in [3], this relation is as follows

$$G^{ab} = \left(\frac{1}{g + 2\pi\alpha'(b + F)} \right)_S^{ab}, \quad (3.5)$$

where the subscript S denotes the symmetric part. Using the explicit expression for the closed string metric and the 2-form fields from section 2, one finds

$$\left(\frac{1}{g + 2\pi\alpha'(b + F)} \right)_{\psi=\psi_n} = \frac{1}{k\alpha'} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\cos \psi_n}{\sin \psi_n} \frac{1}{\sin \theta} \\ 0 & -\frac{\cos \psi_n}{\sin \psi_n} \frac{1}{\sin \theta} & \frac{1}{\sin^2 \theta} \end{pmatrix}. \quad (3.6)$$

Thus, the restriction $P(G)$ of the open string metric on S^2 is $k\alpha'$ times the canonical metric on S^2 . Therefore, the quantities necessary for the computation of the action are obtained as

$$\sqrt{\det P(G)} = k\alpha' \sin \theta, \quad (3.7)$$

and

$$G_s = \frac{g_s}{\sin \frac{\pi n}{k}}, \quad (3.8)$$

where we have substituted the expression for $b + F$ at $\psi = \psi_n$. We see that the relation between open and closed string coupling constants depends on the size n of the fuzzy sphere. This is clearly due to the fact that the NS-NS 2-form field is varying in S^3 . As is discussed in [15], the fuzzy sphere description is only valid in the limit $n/k \ll 1$. In this regime the relation between coupling constants becomes

$$G_s = \frac{g_s k}{\pi n}. \quad (3.9)$$

Having found the relation between the closed and open string coupling constants, it is easy to fix the coefficient in front of the action. To go from the integral over the S^2 to the trace of the fuzzy sphere we have to use the prescription

$$\frac{1}{4\pi k\alpha'} \int \sqrt{P(G)} f \rightarrow \frac{1}{\text{Dim}} \text{Tr} \hat{f}. \quad (3.10)$$

Here f is a function on S^2 and \hat{f} is the corresponding matrix representation on the fuzzy sphere, and Dim is the dimension of the irreducible representation of $\text{SU}(2)$ that is used to construct the fuzzy sphere. The map from the integral to the trace is normalized so that both sides coincide when $f = 1$. Using this prescription we can write

$$S_{\text{YM}} = 4\pi k\alpha' \hat{T}_{(2)} (2\pi\alpha')^2 \frac{1}{\text{Dim}} \text{Tr} \frac{1}{4} F_{ab} F^{ab}. \quad (3.11)$$

Here the indices are raised and lowered using the open string metric G . Using (3.9), we can now rewrite the action in terms of the closed string coupling constant and the flat metric. We get

$$S_{\text{YM}} = nT_{(0)} (2\pi\alpha')^2 \frac{1}{(k\alpha')^2} \frac{1}{\text{Dim}} \text{Tr} \frac{1}{4} F_{ab} F_{ab}. \quad (3.12)$$

This is to be compared with (3.1). We see that what we have obtained is essentially the action (3.1), apart from the prefactor of $nT_{(0)}$. In fact, this prefactor could have been guessed even without the above derivation. Indeed, the n cancels with the factor of $1/\text{Dim}$ normalizing the trace, and what one is left with is just the action for matrix model of D0-branes, with the correct factor of $T_{(0)}$ in front.

Thus, adding the CS term with the same normalization coefficient, we obtain the full expression for the action as

$$S[B] = nT_{(0)} \left(\frac{2\pi}{k} \right)^2 \frac{1}{\text{Dim}} \text{Tr} \left(-\frac{1}{4} [B_a, B_b]^2 + \frac{i}{3} \epsilon_{abc} B_a [B_b, B_c] + \frac{1}{6} Y_a^2 \right), \quad (3.13)$$

Here Y_a satisfy the $SU(2)$ algebra (2.11).

4 Solutions and their D-brane Interpretation

We are now ready to describe the subset of solutions that have an interpretation in terms of D0-branes.

4.1 Subset of solutions

From now on, we will use the action (3.13) written in terms of rescaled covariant coordinates B_a and with all indices contracted with the help of the usual flat metric in \mathbb{R}^3 . The equations of motion that one derives from this action read

$$[B_b, F_{ab}] = 0, \quad (4.1)$$

where the field strength is

$$F_{ab} = i[B_a, B_b] + \epsilon_{abc} B_c. \quad (4.2)$$

A large set of solutions was described in [15] and reviewed in section 2 above. As we said in the introduction, in this paper we would like to consider a particular subset of solutions, which will later receive the interpretation as describing a single spherical D2-brane together with a number of D0-branes. Restriction to a single D2-brane is for simplicity only and can be lifted in a straightforward way. The set of solutions of interest can be constructed by first constructing the one describing branes centered at the origin and then modifying it to shift the branes. A configuration with all branes located at the origin preserves the $SU(2)$ rotational symmetry and, thus, must have the vanishing field strength

$$F_{ab} = 0. \quad (4.3)$$

The definition (4.2) of the field strength shows that this is equivalent to the requirement that B_a are generators of some (not necessarily irreducible) representation of $SU(2)$. The case of B generating an irreducible representation clearly corresponds to a single spherical D2-brane.

The case of B splitting into several irreducible components corresponds to several spherical D2-branes all centered at the origin. Among these irreducible components one can have the trivial representation. It describes D2-branes of zero size, or, equivalently, D0-branes, as we shall demonstrate below.

Hence, a solution describing a single D2-brane plus a number of D0-branes, all centered at the origin is simply

$$B^a = \begin{pmatrix} 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & Y_n^a \end{pmatrix}. \quad (4.4)$$

Here Y_n^a are generators in the irreducible representation of dimension $(n+1)$. To get a more general solution with arbitrary brane locations we need to shift their positions. We can shift both the D2-brane and D0-branes, but this is not necessary since we are only interested in relative configuration of the branes. Thus, we will always keep the D2-brane centered at the origin. Then the shifting of D0-branes amounts to replacing zero's on the diagonal with numbers that receive the interpretation of coordinates of D0-branes in S^3 :

$$B^a = \begin{pmatrix} c_1^a & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & c_l^a & 0 \\ 0 & 0 & 0 & Y_n^a \end{pmatrix}. \quad (4.5)$$

The number of D0-branes described is equal to l . It is easy to see that the field strength (4.2) on (4.5) is non-zero and is equal to

$$F_{ab} = \epsilon_{abc} \begin{pmatrix} c_1^c & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & 0 \\ 0 & \dots & c_l^c & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.6)$$

This clearly satisfies the field equations, for F_{ab} commutes with B^a . Thus, (4.5) is indeed a solution.

Before we proceed with the discussion as to the properties of this solution, several comments are in order. First of all, let us note that, unlike the discussion in [15], we choose to describe the solution directly in terms of the covariant coordinates B^a , not in terms of the gauge field A^a . This is more convenient, for the components of the covariant coordinates B^a receive the direct interpretation of coordinates in S^3 , in the case when B can be (at least partially) diagonalized. In addition, this description in terms of B^a is rather natural, because

it makes transparent the relation to T-dualities in Matrix theory [23, 24]. Second, the set of solutions that is obtainable from D-branes centered at the origin by shifts by commuting matrices is, probably, not the most general set. This way one can only obtain configurations solving (4.1) rather trivially, in the sense that F commutes with B . It would be interesting to find examples of solutions, if any, for which this is not the case.

The solution (4.5) is the direct analog of the “unstable” solitons found in the context of flat D2-branes, see [7, 8]. As is explained in [8], in non-commutative gauge theory on flat D2-brane, the requirement of finiteness of energy only allows configurations in which the covariant derivative D (and its conjugate \bar{D}) generates a reducible representation of the Heisenberg algebra. The covariant derivative D , or, in the terminology of [25] covariant coordinate, is an infinite rank matrix. The requirement of finite energy fixes it to be of a block diagonal form, in which the first block is diagonal, with numbers on the diagonal having the interpretation of positions of solitons, and the second block filling the usual infinite dimensional representation of the Heisenberg algebra. Our solution (4.5) falls into the same general scheme: it forms a reducible representation of the symmetry algebra in question, splitting into a single irreducible and several trivial components. Like in the flat case, the irreducible component describes the D2-brane itself, and trivial representations correspond to D0-branes. The main difference with the flat case is that the D2-brane is represented by a finite dimensional matrix. Note that one can add other non-trivial irreducible representations. This means that we can have small spherical D2-branes as “solitons” of the large mother D2-brane, although we do not consider this case in the present paper. Note that the distinction of which D2-brane is a “soliton” and which provides a “background” becomes relative. Any D2-brane can be viewed as the “mother” on whose worldvolume the non-commutative gauge theory lives. This serves as a good illustration of the “brane democracy” [9] in string theory.

Therefore, our solution is the analog of the one found in [7, 8], and thus should also describe D0-branes. We shall directly confirm this interpretation by comparing the energy found in the ordinary BI and non-commutative descriptions. However, before we proceed with this comparison of energies, let us make one more remark. As we have seen, the configuration corresponding to a single D2-brane has vanishing field strength F_{ab} (this also follows from its spherical symmetry). This is a little puzzling, for the integral of this two form over the D2-brane has the interpretation of the number of D0-branes. Indeed, the intuition based on the Myers effect [20] is that the fuzzy D2-brane worldvolume is “made out” of a number of D0-branes. However, by simply integrating the non-commutative field strength over the D2-brane, which in non-commutative description is replaced by the operation of taking the trace, we get zero. The resolution of this puzzle is that in non-commutative description the number of D0-branes is no longer given by the integral (trace) of F . This

interpretation comes in the commutative case from the CS term

$$T_p \int d^{p+1}x \left(\sum C \right) \wedge e^{2\pi\alpha'(b+F)}. \quad (4.7)$$

As it has been discussed in the literature, see, e.g., [26], in the non-commutative description the quantity $2\pi\alpha'(b+F)$ in the exponential must be replaced by its Seiberg-Witten map [3] image, which is the matrix that was denoted by Q^{-1} in [26]. One also gets a factor of $\text{Pf}Q$ under the trace, see [26]. As one can easily check, this gives that the number of D0-branes is given in non-commutative description by the trace of the identity operator, or by the dimension of the Hilbert space. This is of course consistent with what we expect from the Myers effect interpretation of the fuzzy D2-brane, for the D2-brane made out of n D0-branes is described by matrices of rank n . In the flat case, the non-commutative D2-brane can be thought of as made of an infinite number of D0-branes, and the operator $\text{Tr}F$ measures only the number of not yet dissolved D0-branes, not the total number in the system. It is instructive to compare the discussion of this paragraph to the discussion on the quantization of D0-brane charge appeared recently in the literature, see, e.g., [12, 27, 28].

4.2 Comparison of the energy

In order to confirm that the solution (4.5) actually corresponds to a set of the D0-branes and a spherical D2-brane we compare the energy of this solution to the energy that can be obtained in the ordinary BI description of section 2. First, let us obtain the prediction of the ordinary BI theory.

We would like to understand the n dependence of the energy of the spherical D2-brane (2.8) in the region $n/k \ll 1$. Physically, this corresponds to D2-branes of radius much smaller than the radius of curvature of S^3 . It is for such branes that the our description in terms of the gauge theory on a fuzzy sphere is applicable. We have

$$E_n = 4\pi k \alpha' T_{(2)} \left[\left(\frac{\pi n}{k} \right) - \frac{1}{3!} \left(\frac{\pi n}{k} \right)^3 \right] + \mathcal{O}((n/k)^5) = T_{(0)} n \left[1 - \frac{1}{3!} \left(\frac{\pi n}{k} \right)^2 \right] + \mathcal{O}((n/k)^5), \quad (4.8)$$

where we have used the fact that $4\pi^2 \alpha' T_{(2)} = T_{(0)}$. The first term has an obvious interpretation as the energy of n D0-branes. The second term can be interpreted as the binding energy which is released when the n D0-branes form a single D2-brane. Thus, in the limit $n/k \ll 1$ we can write

$$E_{n\text{D0}} - E_n = T_{(0)} n \frac{1}{3!} \left(\frac{\pi n}{k} \right)^2. \quad (4.9)$$

As we shall see in a moment, this energy difference is reproduced by the non-commutative description.

Another quantity that we need is the binding energy of a single D0-brane and a D2-brane. This can be derived analogously to what was done in Ref. [7]. One expands the BI action in the limit of large $b + F$ field. The binding energy is then read from the variation of the total energy when $b + F$ changes so that

$$\int \delta(b + F) = 2\pi, \quad (4.10)$$

which corresponds to the emission of a single D0-brane. Completely analogously to the flat case, we get

$$E_{\text{bind}} = \frac{T_{(0)}}{2} \left(\frac{\sqrt{g}}{2\pi\alpha'(b + F)} \right). \quad (4.11)$$

Here \sqrt{g} is the square root of the determinant of the induced (closed string) metric on the D2-brane, and $b + F$ is the $(\theta\varphi)$ -component of the $b + F$ two-form. Substituting the values of these quantities at ψ_n we get

$$E_{\text{bind}} = \frac{T_{(0)}}{2} \tan^2 \frac{\pi n}{k} \approx \frac{T_{(0)}}{2} \left(\frac{\pi n}{k} \right)^2, \quad (4.12)$$

where the last quantity is in the limit $n/k \ll 1$.

Let us now calculate the same quantities in the non-commutative description. First the energy of n D0-branes not yet forming a D2-brane can be obtained by evaluating the action on $B_a = 0$ and is given by

$$nT_{(0)} \frac{(2\pi\alpha')^2}{(k\alpha')^2} \frac{1}{\text{Dim}} \text{Tr} \frac{1}{6} (Y)^2. \quad (4.13)$$

In the n -dimensional irreducible representation that we work with, one can replace $(Y)^2$ with the value of the Casimir $(n^2 - 1)/4$. We get

$$nT_{(0)} \frac{1}{6} \frac{\pi^2(n^2 - 1)}{k^2}. \quad (4.14)$$

The energy of a single D2-brane formed out of n D0-branes can be obtained by evaluating the action on $B^a = Y_{n-1}^a$, where Y_{n-1}^a are generators in the n -dimensional irreducible representation. This gives zero. Thus, in the non-commutative description the zero of energy is taken to be at the bound state of n D0-branes forming a D2-brane. The quantity (4.14) is then to be compared with the energy difference (4.9). In the limit $n/k \ll 1$ and $n \gg 1$ ensuring that the ordinary BI description is acceptable, the two agree. Note that the factor of $1/6$ in the non-commutative action receives the interpretation of $1/3!$ coming from the sin in the ordinary BI result.

The second quantity that must be calculated is the binding energy. It is to be compared to the binding energy (4.12). Let us however calculate a more general quantity, namely the

energy of the solution (4.5) describing l D0-branes. We have

$$\begin{aligned}
& (n+l)T_{(0)} \frac{(2\pi\alpha')^2}{(k\alpha')^2} \frac{1}{n+l} \text{Tr} \frac{1}{6} ((Y_{n+l-1})^2 - (Y_{n-1})^2) \\
&= (n+l)T_{(0)} \frac{\pi^2}{k^2} \frac{1}{6} \left((n+l)^2 - 1 - (n^2 - 1) \frac{n}{n+l} \right) \\
&\approx \frac{lT_{(0)}}{2} \left(\frac{\pi n}{k} \right)^2,
\end{aligned} \tag{4.15}$$

which is exactly the number of D0-branes l times the BI result (4.12) for the binding energy, as one expects. We thus showed the perfect matching between the energies computed in the non-commutative description and the expectations from the BI theory.

5 Fluctuations and Stability

As we argued in the previous section, the solutions considered have the interpretation in terms of D0-branes. In the flat space, a D0-brane located close enough to a (flat) D2-brane is known to be unstable which is manifested by the presence of a tachyonic mode in the string spectrum. The final stable configuration in this case is the D0-brane absorbed by the D2-brane, that is, the D0-D2 bound state. We expect a similar phenomenon to be present in our case. One can see whether or not a tachyonic mode is present by studying small fluctuations around the solution in question. Such a study was performed in the flat case in [7], and here we present a similar analysis for our case.

To study fluctuations, we need an expression for the second variation of the action. A simple computation shows that it is given by

$$\delta^2 S = \text{Tr} \left(-[\delta B_a, B_b][\delta B_a, B_b] + ([\delta B_a, B_a])^2 + 2i[\delta B_a, \delta B_b]F_{ab} \right). \tag{5.1}$$

This expression generalizes the one given in [15] to the case of $F \neq 0$.

We would like to study fluctuations around the soliton describing a single D0-brane plus a spherical D2-brane. One can always put the center of the D2-brane to the origin of S^3 . The corresponding solution then is given by

$$B_a = \begin{pmatrix} c_a & \vec{0}^T \\ \vec{0} & Y_a \end{pmatrix} \tag{5.2}$$

in the matrix notation. Here Y_a are SU(2) generators in the $n+1$ -dimensional irreducible representation, that is, the representation of spin $n/2$. Thus, we are describing a D2-brane made out of $(n+1)$ D0-branes here. The constant c_a parameterizes the moduli of the solution,

and has the interpretation of a position of the D0-brane. Note that c_a is a real vector due to the Hermiticity of the matrix B^a . The spherical D2-brane, represented by the part Y_a in the above matrix, is a spherical shell of a definite radius, depending on n . We expect that the above soliton has a tachyonic fluctuation in a certain limited region of moduli space of c_a . This region should corresponds to the situation that the D0-brane sits very close to the shell of the D2-brane.

Experience with the flat case teaches us that the tachyonic mode comes from the excitations of the string connecting the D0-brane and D2-branes. As in the works [5, 6, 7] on the spectrum of fluctuations around non-commutative solitons in flat case, the 0-2 string modes are described by the off-diagonal elements of the matrix representing the soliton. Hoping that the same is true in our case, let us turn on only the off-diagonal fluctuations. Thus, we consider the following perturbation of the solution

$$\delta B_a = \begin{pmatrix} 0 & v_a^\dagger \\ v_a & 0 \end{pmatrix} \quad (5.3)$$

where v_a is a complex vectors (column) of dimension $n + 1$. Substituting this fluctuation mode into the action, we get

$$\delta^2 S = v_a^\dagger (Y_b - c_b \mathbb{I}) (Y_b - c_b \mathbb{I}) v_a - v_a^\dagger (Y_a - c_a \mathbb{I}) (Y_b - c_b \mathbb{I}) v_b + i \epsilon_{abc} c_a (v_b^\dagger v_c - v_c^\dagger v_b) \quad (5.4)$$

We can use the rotation symmetry on S^3 to fix the value of c_a as $c_a = \delta_{a3} c$, without losing generality. The above expression is a Hermitian quadratic form in the complex vector space of dimension $3(n + 1)$ (3 complex vectors v_a). To facilitate its diagonalization, let us define the following complex linear combinations of these 3 vectors

$$\mathbf{v} = v_3, \quad \mathbf{v}_+ = \frac{1}{\sqrt{2}}(v_1 + iv_2), \quad \mathbf{v}_- = -\frac{1}{\sqrt{2}}(v_1 - iv_2). \quad (5.5)$$

The minus in front of the expression for \mathbf{v}_- is for uniformity with the similar definition of Y_- , see Appendix. Let us at the same time introduce the usual complex linear combinations of SU(2) generators, see the Appendix for our conventions on normalization. We shall denote these “raising and lowering” operators by Y, Y_\pm . The action, written in terms of these quantities, becomes

$$\begin{aligned} \frac{1}{4} \delta^2 S &= \left(\mathbf{v}^\dagger \mathbf{v} + \mathbf{v}_+^\dagger \mathbf{v}_+ + \mathbf{v}_-^\dagger \mathbf{v}_- \right) \left(\frac{n}{2} \left(\frac{n}{2} + 1 \right) + c^2 \right) - 2c \left(\mathbf{v}^\dagger Y \mathbf{v} + \mathbf{v}_+^\dagger Y \mathbf{v}_+ + \mathbf{v}_-^\dagger Y \mathbf{v}_- \right) \\ &- \left(\mathbf{v}^\dagger Y + \mathbf{v}_+^\dagger Y_+ + \mathbf{v}_-^\dagger Y_- \right) (Y \mathbf{v} - Y_+ \mathbf{v}_- - Y_- \mathbf{v}_+) \\ &+ c \left(\mathbf{v}^\dagger Y + \mathbf{v}_+^\dagger Y_+ + \mathbf{v}_-^\dagger Y_- \right) \mathbf{v} + c \mathbf{v}^\dagger (Y \mathbf{v} - Y_+ \mathbf{v}_- - Y_- \mathbf{v}_+) - c^2 \mathbf{v}^\dagger \mathbf{v} \\ &+ 2c \left(\mathbf{v}_+^\dagger \mathbf{v}_+ - \mathbf{v}_-^\dagger \mathbf{v}_- \right). \end{aligned} \quad (5.6)$$

Here, to write the first term, we have used the fact that the $(n+1)$ -dimensional complex vector space that the vectors $\mathbf{v}, \mathbf{v}_\pm$ are elements of can be thought of as the irreducible representation space. Thus, the operator $Y_a Y_a$ can be replaced by the value of the Casimir times the identity operator. To diagonalize this quadratic form, let us introduce the basis of eigenvectors $|m\rangle$ of the operator Y , and decompose the vectors $\mathbf{v}, \mathbf{v}_\pm$ with respect to this basis. Using conjugation of generators Y, Y_\pm by elements of $U(n+1)$ we can always choose the highest vector $|n/2\rangle$ to point along \mathbf{v} . The decomposition then becomes:

$$\mathbf{v} = v |n/2\rangle, \quad \mathbf{v}_+ = \sum_m v_+^m |m\rangle, \quad \mathbf{v}_- = \sum_m v_-^m |m\rangle. \quad (5.7)$$

Substituting these decompositions into (5.6), and using the standard expressions for the action of Y_\pm on vectors $|m\rangle$, see Appendix, one can diagonalize the quadratic form in question rather straightforwardly. The action takes a block diagonal form consisting of blocks not larger than 2×2 . The coupling of the modes to each other is as follows. The mode v gets coupled only to $v_-^{n/2-1}$. This part of the action is

$$|v|^2 \left(\frac{n}{2}\right) + \bar{v} v_-^{n/2-1} \sqrt{\frac{n}{2}} \left(\frac{n}{2} - c\right) + c.c. + |v_-^{n/2-1}|^2 \left(\frac{n}{2} - c\right)^2. \quad (5.8)$$

This quadratic form can be easily diagonalized with the eigenvalues being

$$\lambda_0 = 0, \quad \lambda_+^{n/2} = c^2 - nc + \frac{n}{2} \left(\frac{n}{2} + 1\right). \quad (5.9)$$

The notation for the last eigenvalue will become clear when we consider the other modes. The modes $v_-^{n/2}$ and $v_+^{-n/2}, v_+^{-n/2+1}$ do not couple to other modes. We get for each of these modes correspondingly

$$|v_-^{n/2}|^2 \left(c^2 - 2c \left(\frac{n}{2} + 1\right) + \frac{n}{2} \left(\frac{n}{2} + 1\right)\right), \quad (5.10)$$

$$|v_+^{-n/2}|^2 \left(c^2 + 2c \left(\frac{n}{2} + 1\right) + \frac{n}{2} \left(\frac{n}{2} + 1\right)\right), \quad (5.11)$$

$$|v_+^{-n/2+1}|^2 \left(\frac{n}{2} + c\right)^2. \quad (5.12)$$

Thus, the corresponding eigenvalues are

$$\lambda_t = c^2 - 2c \left(\frac{n}{2} + 1\right) + \frac{n}{2} \left(\frac{n}{2} + 1\right), \quad (5.13)$$

$$\tilde{\lambda}_t = c^2 + 2c \left(\frac{n}{2} + 1\right) + \frac{n}{2} \left(\frac{n}{2} + 1\right), \quad (5.14)$$

$$\lambda_-^{-n/2} = \left(\frac{n}{2} + c\right)^2. \quad (5.15)$$

We have introduced a special notation for the first two eigenvalues because, as we shall see, they correspond to tachyonic modes. The remainder of the action couples, in pairs, the

modes v_+^{m+1} and v_-^{m-1} , with $m = (n/2 - 1), (n/2 - 2), \dots, (-n/2 + 1)$:

$$\begin{aligned}
& |v_+^{m+1}|^2 \left(c^2 - 2cm + \frac{1}{2}m(m+1) + \frac{1}{2} \left(\frac{n}{2} \right) \left(\frac{n}{2} + 1 \right) \right) \\
& + \bar{v}_+^{m+1} v_-^{m-1} \frac{1}{2} \sqrt{\left(\left(\frac{n}{2} + 1 \right)^2 - m^2 \right) \left(\left(\frac{n}{2} \right)^2 - m^2 \right)} + c.c. \\
& + |v_-^{m-1}|^2 \left(c^2 - 2cm + \frac{1}{2}m(m-1) + \frac{1}{2} \left(\frac{n}{2} \right) \left(\frac{n}{2} + 1 \right) \right).
\end{aligned} \tag{5.16}$$

The corresponding eigenvalues are

$$\lambda_-^m = (c - m)^2, \quad \lambda_+^m = c^2 - 2mc + \frac{n}{2} \left(\frac{n}{2} + 1 \right). \tag{5.17}$$

Here we have introduced the notation λ_{\pm} for two different sets of eigenvalues. To summarize, the set of eigenvalues consists of: (i) $\lambda_t, \tilde{\lambda}_t$ which correspond to tachyonic modes; (ii) zero mode λ_0 ; (iii) two sets λ_{\pm}^m , with $m = n/2, n/2 - 1, \dots, -n/2 + 1$ for λ_+^m (no $m = -n/2$) and $m = n/2 - 1, \dots, -n/2 + 1, -n/2$ for λ_-^m (no $m = n/2$).

As we have said, the eigenvalues $\lambda_t, \tilde{\lambda}_t$ correspond to tachyonic modes. Indeed, in the region of the moduli space

$$c \in \left(\frac{n}{2} + 1 - \sqrt{\frac{n}{2} + 1}, \frac{n}{2} + 1 + \sqrt{\frac{n}{2} + 1} \right) \tag{5.18}$$

the eigenvalue λ_t is negative, and in the region

$$c \in \left(-\left(\frac{n}{2} + 1 \right) - \sqrt{\frac{n}{2} + 1}, -\left(\frac{n}{2} + 1 \right) + \sqrt{\frac{n}{2} + 1} \right) \tag{5.19}$$

the eigenvalue $\tilde{\lambda}_t$ is negative. These values of c correspond, as expected, to two intervals on the z -axes, near the intersection of the axes with the spherical D2-brane shell. Indeed, let us convert the distances measured by c into physical distances measured with respect to the closed string metric g . The conversion is

$$(\text{distance}) = 2\pi\alpha' \frac{c}{\sqrt{k\alpha'}}. \tag{5.20}$$

The region of instability is then centered around the value $c = \pm(n/2 + 1)$, which corresponds, in physical units, to the points the distance

$$R = \pi(n + 2) \sqrt{\frac{\alpha'}{k}} \tag{5.21}$$

away from the origin. This is the correct radius of the fuzzy D2-brane, as predicted by the commutative description

$$R = \sqrt{\alpha' k} \sin \frac{\pi n}{k} \approx \pi n \sqrt{\frac{\alpha'}{k}}. \tag{5.22}$$

As far as the instability is concerned, it appears at the physical distance of

$$L_{\text{instab}} = (2\pi R)^{1/2} \left(\frac{\alpha'}{k} \right)^{1/4} \quad (5.23)$$

away from the shell. Here we have used the radius of the shell R so as to eliminate the n dependence. One would like to compare this with the flat case result $L_{\text{instab}} \sim \sqrt{\alpha'}$. We see that the result (5.23) is *different* from the flat result in the regime $R \ll \sqrt{k\alpha'}$ where one can neglect the curvature of S^3 and the fuzzy sphere description is valid. If, for some reason, the result (5.23) can be extrapolated beyond its region of validity into the region $R \sim \sqrt{k\alpha'}$ of radii of S^2 being of the order of the radius of curvature of S^3 , then L_{instab} equals to the flat case quantity for $R = \sqrt{k\alpha'}$.

Note that each tachyonic eigenvalue λ_t or $\tilde{\lambda}_t$ represents one complex, or equivalently two real tachyonic modes. These have the interpretation of coming from the two different orientations of the 0-2 string.

Let us now discuss the interpretation of other eigenvalues. The zero mode appears because some symmetry is left unbroken in the solution under consideration. Indeed, we still have a $U(1)$ subgroup of the global $SU(2)$ rotating the generators. This subgroup generates rotations about the axes of symmetry (z) of our solution. There is, in addition, another $U(1)$, which is a subgroup of $U(n+2)$ acting on our matrices by conjugation. This $U(1)$ multiplies the rank 1 and $(n+1)$ blocks of the solution by the complex conjugate phase factors. These two $U(1)$ correspond to two real zero modes in the fluctuation spectrum.

All other eigenvalues are positive for any c , as can be easily checked. The corresponding masses thus constitute a prediction for the low energy spectrum of 0-2 worldsheet CFT on S^3 in the SW limit. In the flat case, these spectrum can be easily calculated on the CFT side, see [7], and one finds an exact agreement with the non-commutative prediction. It would be somewhat harder to do a similar CFT calculation in our case of S^3 , and we shall not attempt it here.

6 Higgs Mechanism

Our results on stability have potentially interesting implications as to scenarios of gauge symmetry breaking. Let us assume that the manifold M_7 , which is transverse to S^3 , contains some flat directions, i.e., is of the type $\mathbb{R}^{p+1} \times \widetilde{M}_{6-p}$. Let us have a stack of N Dp -branes, such that the directions along the branes are in \mathbb{R}^p . One can then get an unusual gauge symmetry breaking mechanism by separating the branes in S^3 .

The low energy excitations of a stack of Dp-branes in \mathbb{R}^{10} are described by the usual action

$$S = T_{(p)}(2\pi\alpha')^2 \int d^{p+1}x \text{Tr} \left(\frac{1}{4} f_{\mu\nu}^2 + \frac{1}{2} (D_\mu X_a)^2 - \frac{1}{4} [X_a, X_b]^2 \right). \quad (6.1)$$

Here $f_{\mu\nu}$ is the curvature of the connection a_μ , with Greek indices corresponding to directions along the branes, and X_a are transverse scalars. When one puts the branes on $S^3 \times \mathbb{R}^{p+1} \times \widetilde{M}_{6-p}$ as described, the last term in this action must be replaced by the action (3.13) of the gauge theory on fuzzy sphere, with the identification

$$X_a = B_a / \sqrt{k\alpha'}. \quad (6.2)$$

There is also the commutator term for other transverse coordinates when $p < 6$, but we will not consider it. We thus get

$$S = T_{(p)}(2\pi\alpha')^2 \int d^{p+1}x \text{Tr} \left(\frac{1}{4} f_{\mu\nu}^2 + \frac{1}{2} \frac{1}{k\alpha'} (D_\mu B_a)^2 \right) + \int d^{p+1}x S[B], \quad (6.3)$$

where $T_{(0)}$ in $S[B]$ must be replaced with $T_{(p)}$.

Let us first analyze the case of two Dp-branes, when all fields are 2×2 matrices. As we know, there are two main configurations which locally minimizes $S[B]$ in this case: one corresponding to branes simply separated in S^3 (commuting B^a), and the other corresponding to a fuzzy sphere. In the first case one can take only B_3 to be non-zero, and given by $B_3 = c\sigma_3/2$. This gives the usual D-brane Higgs mechanism. For such B_a , the second term in the action breaks symmetry down to $U(1)$. Decomposing $a_\mu = \sum_i a_\mu^i \sigma_i / \sqrt{2}$ we get the masses of the off-diagonal gauge bosons a_μ^1, a_μ^2 to be

$$m = \frac{c}{\sqrt{k\alpha'}}. \quad (6.4)$$

Converting c into the physical distance between the D0-branes as measured with respect to the closed string metric $L = 2\pi\alpha'c/\sqrt{k\alpha'}$, we get

$$m = \frac{L}{2\pi\alpha'}. \quad (6.5)$$

This is, of course, just the energy of a fundamental string of length L , as expected.

However, for small L this mechanism should be modified. Indeed, as D0-branes become too close to each other, the configuration becomes unstable, and the fuzzy sphere forms. The value of c for which this happens can be found from the results of the previous section. For this one must substitute $n = 0$ in the formula for the tachyonic eigenvalues. This corresponds to a system of two D0-branes. We then get that the distance between the D0-branes (in the

unphysical metric) when the instability occurs is 2. Thus, one must take $c = 2$ in (6.4). This corresponds to the physical distance between the branes

$$L_{\text{instab}} = \frac{4\pi\alpha'}{\sqrt{k\alpha'}} \quad (6.6)$$

and the mass

$$m_{\text{instab}} = \frac{2}{\sqrt{k\alpha'}}. \quad (6.7)$$

This is the smallest mass of the off-diagonal gauge bosons that the a_μ^3 boson is still massless. As one further decreases the distance between the branes the fuzzy sphere forms which breaks the gauge symmetry completely. It is easy to find the masses of gauge bosons after the symmetry is broken. They are again obtained from the second term in the action (6.3), evaluated on the configuration $B_a = Y_a = \sigma_a/2$. Using the same decomposition of the connection into Pauli matrices as above, we get all three masses to be equal to

$$m = \frac{\sqrt{2}}{\sqrt{k\alpha'}}. \quad (6.8)$$

This is by $\sqrt{2}$ smaller than the smallest mass m_{instab} that can be achieved in the usual Higgs mechanism. One can say that some of the mass of a^1, a^2 went into a^3 .

Thus, there is the smallest mass (6.7) that can be achieved by the usual Higgs mechanism. When one tries to further decrease it decreasing the separation between the D0-branes the system becomes unstable and forms the fuzzy sphere, which breaks the gauge symmetry completely. All three gauge bosons become massive with the mass given by (6.8). Note interestingly, while the local gauge symmetry is broken completely, the global symmetry is enhanced to the full $SU(2)$ as compared to the global symmetry of $U(1)$ in the usual Higgs breaking mechanism. Thus, summarizing, if the transverse coordinates form S^3 , there is no usual restoration of gauge symmetry by decreasing the separation between the branes. The only way to restore the gauge symmetry is to take the flat space limit $\sqrt{k\alpha'} \rightarrow \infty$.

Other symmetry breaking patterns can be obtained by considering a fuzzy sphere of size n plus a single D0-brane. When the D0-brane is far enough from the shell, the $U(n+1)$ symmetry of the system is broken to $U(1)$. As one decreases the distance between the shell and the D0-brane, the system becomes unstable and the fuzzy sphere of larger radius forms. The gauge symmetry is then broken completely. Masses of gauge bosons before and after the formation of the fuzzy sphere can be found similarly to the above calculation.

7 Conclusions and Discussion

In this paper we identified a simple subset of solutions of gauge theory on fuzzy sphere and gave their interpretation in terms of D0-branes located off a D2-brane. We confirmed this interpretation by comparing the energy obtained in non-commutative and commutative descriptions. We also looked at the spectrum of fluctuations about the simplest solution containing a single D2-brane and D0-branes. We have found a tachyonic mode when the D0-brane is located close enough to the shell of the D2-brane. This is similar to the flat case, although the expression for the distance when the instability occurs depends on the radius of the D2-brane shell and the radius of S^3 and is thus different from the flat case result. We have also discussed how this instability modifies the usual Higgs mechanism for small separation between the branes.

Since the instability occurs when D0-brane is too close to the shell of D2-brane, it is natural to expect that a similar instability is present in a system of several non-cocentric D2-branes, in the case when the shells are too close to each other. However, this case would be much harder to analyze quantitatively, and we did not consider it in this paper.

Considering the fluctuations we have analyzed only the 0-2 part of the spectrum, which is the most interesting because it is this part that contains the tachyonic modes. It would be interesting to compare our prediction for the 0-2 spectrum with a direct CFT calculation for 0-2 string in S^3 in the SW limit. Another possible calculation is to consider 0-0 and 2-2 modes. For the later case there is a prediction from the usual BI theory in [12]. These fluctuations in the non-commutative description were recently studied in [22], but no diagonalization was given. Thus, it is still an open problem to match the non-commutative spectrum of 2-2 fluctuations to the BI prediction of [12].

As is discussed in [15], and more recently in [22], the non-commutative gauge theory action we considered is a bosonic part of certain supersymmetric theory on the fuzzy sphere. Although the SUSY transformation law for fermions was not given in [15], it can be guessed by using the fact that the action is the sum of YM and CS terms. Indeed, for the case of commutative (but not necessarily Abelian) YM CS gauge theory, the supersymmetry transformation for the gaugino is [29, 30]

$$\delta\lambda = \frac{1}{2}F_{ab}\gamma^{ab}\eta \tag{7.1}$$

where η is an infinitesimal Dirac spinor parameter, F_{ab} are the field strength, and γ^{ab} are the usual commutators of 3D γ -matrices. A natural way to generalize it to the non-commutative case is to replace the field strength F by its non-commutative version (4.2). This would be consistent with the claim of [10, 11, 12] that spherical D2-branes constitute supersymmetric

configurations, for the field strength for them vanishes, and the gaugino is SUSY invariant*. However, the transformation law (7.1) with F given by (4.2) is not what is claimed in [22] to be the correct law for SUSY gauge theory on the fuzzy sphere. There, the authors had proposed a SUSY extension of the bosonic action with the transformation law using the usual $F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]$. This field strength does not vanish for a fuzzy sphere, which makes it not a SUSY configuration. A possible resolution of this apparent contradiction might be that there are several different SUSY extensions of the bosonic action (3.13), and in the context of D-branes the one with the transformation law (7.1) with F given by (4.2) should be used. Such a SUSY extension is not the one given in [22], and it remains to be seen whether it exists.

A SU(2) operators

We use the following definition of “raising and lowering” SU(2) operators:

$$Y = Y_3, \quad Y_+ = \frac{1}{\sqrt{2}}(Y_1 + iY_2), \quad Y_- = -\frac{1}{\sqrt{2}}(Y_1 - iY_2). \quad (\text{A.1})$$

Our conventions are the same as those of [33]. The action of Y_{\pm} on eigenvectors of Y , in the $(n+1)$ -dimensional irreducible representation, is

$$Y_+|m\rangle = \sqrt{\frac{1}{2}\left(\frac{n}{2} - m\right)\left(\frac{n}{2} + m + 1\right)}|m+1\rangle, \quad (\text{A.2})$$

$$Y_-|m\rangle = -\sqrt{\frac{1}{2}\left(\frac{n}{2} + m\right)\left(\frac{n}{2} - m + 1\right)}|m-1\rangle. \quad (\text{A.3})$$

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*Although the translation of the spherical D2-brane in S^3 shifts F to a non-zero value and thus seems to break the supersymmetry, this could be restored by introducing non-linearly realized supersymmetries that are usually present in the D-brane actions [31, 32].

References

- [1] A. Connes, M. R. Douglas and A. Schwarz, “Noncommutative geometry and matrix theory: compactification on tori”, *JHEP* **9802** (1998) 003, [hep-th/9711162](#).
- [2] M. R. Douglas and C. Hull, “D-branes and the noncommutative torus”, *JHEP* **9802** (1998) 008, [hep-th/9711165](#).
- [3] N. Seiberg and E. Witten, “String theory and noncommutative geometry”, *JHEP* **9909** (1999) 032, [hep-th/9908142](#).
- [4] J. A. Harvey, P. Kraus, F. Larsen and E. J. Martinec, “D-branes and Strings as Noncommutative Solitons”, *JHEP* **0007** (2000) 042, [hep-th/0005031](#).
- [5] D. J. Gross and N. Nekrasov, “Monopoles and strings in noncommutative gauge theory”, *JHEP* **0007** (2000) 034, [hep-th/0005204](#).
- [6] D. J. Gross and N. Nekrasov, “Dynamics of strings in noncommutative gauge theory”, *JHEP* **0010** (2000) 021, [hep-th/0007204](#).
- [7] M. Aganagic, R. Gopakumar, S. Minwalla and A. Strominger, “Unstable Solitons in Noncommutative Gauge Theory”, [hep-th/0009142](#).
- [8] D. J. Gross and N. Nekrasov, “Solitons in noncommutative gauge theory”, [hep-th/0010090](#).
- [9] P. K. Townsend, “p-brane Democracy”, Proceedings in PASCOS/Hopkins conference (1995) 271, [hep-th/9507048](#).
- [10] A. Y. Alekseev and V. Schomerus, “D-branes in the WZW model”, *Phys. Rev.* **D60** (1999) 061901, [hep-th/9812193](#).
- [11] A. Y. Alekseev, A. Recknagel and V. Schomerus, “Noncommutative world volume geometries: branes on SU(2) and fuzzy spheres”, *JHEP* **9909** (1999) 023, [hep-th/9908040](#).
- [12] C. Bachas, M. Douglas and C. Schweigert, “Flux Stabilization of D-branes”, *JHEP* **0005** (2000) 048, [hep-th/0003037](#).
- [13] J. Pawelczyk, “SU(2) WZW D-branes and their noncommutative geometry from DBI action”, *JHEP* **0008** (2000) 006, [hep-th/0003057](#).
- [14] J. Madore, “The fuzzy sphere”, *Class. Quant. Grav.* **9** (1992) 69.

- [15] A. Y. Alekseev, A. Recknagel and V. Schomerus, “*Brane Dynamics in Background Fluxes and noncommutative Geometry*”, *JHEP* **0005** (2000) 010, [hep-th/0003187](#).
- [16] T. Kitao, K. Ohta and N. Ohta, “*Three-Dimensional Gauge Dynamics from Brane Configurations with (p, q) -Fivebrane*”, *Nucl. Phys.* **B539** (1999) 79, [hep-th/9808111](#).
- [17] Y. Imamura, “*A D2-brane realization of Maxwell-Chern-Simons-Higgs systems*”, [hep-th/0012254](#).
- [18] E. Witten, “*Bound states of string and p -branes*”, *Nucl. Phys.* **B460** (1996) 335, [hep-th/9510135](#).
- [19] R. Emparan, “*Born-Infeld Strings Tunneling to D-branes*”, *Phys. Lett.* **B423** (1998) 71, [hep-th/9711106](#).
- [20] R. C. Myers, “*Dielectric-branes*”, *JHEP* **9912** (1999) 022, [hep-th/9910053](#).
- [21] D. K. Park, S. Tamaryan and H. J. W. Müller-Kirsten, “*The Stability of D2-Branes in the Presence of an RR field*”, [hep-th/0012108](#).
- [22] S. Iso, Y. Kimura, K. Tanaka and K. Wakatsuki, “*Noncommutative gauge theory on fuzzy sphere from matrix model*”, [hep-th/0101102](#).
- [23] W. Taylor, “*D-brane field theory on compact spaces*”, *Phys. Lett.* **B394** (1997) 283, [hep-th/9611042](#).
- [24] T. Kawano and K. Okuyama, “*Matrix Theory on Noncommutative Torus*”, *Phys. Lett.* **B433** (1998) 29, [hep-th/9803044](#).
- [25] J. Madore, S. Schraml, P. Schupp and J. Wess, “*Gauge Theory on Noncommutative Spaces*”, *Eur. Phys. J.* **C16** (2000) 161, [hep-th/0001203](#).
- [26] S. Mukhi and N. V. Suryanarayana, “*Chern-Simons terms on noncommutative branes*”, *JHEP* **0011** (2000) 006, [hep-th/0009101](#).
- [27] W. Taylor, “*D2-branes in B fields*”, *JHEP* **0007** (2000) 039, [hep-th/0004141](#).
- [28] D. Marolf, “*Chern-Simons terms and the three notions of charge*”, [hep-th/0006117](#).
- [29] H. Kao, K. Lee and T. Lee, “*The Chern-Simons coefficient in supersymmetric Yang-Mills Chern-Simons theories*”, *Phys. Lett.* **B373** (1996) 94, [hep-th/9506170](#).
- [30] N. Maggiore, O. Piguet and M. Ribordy, “*Algebraic renormalization of $N = 2$ supersymmetric Yang-Mills Chern-Simons theory in the Wess-Zumino gauge*”, *Helv. Phys. Acta* **68** (1995) 264, [hep-th/9504065](#).

- [31] J. Bagger and A. Galperin, “*New Goldstone multiplet for partially broken supersymmetry*”, *Phys. Rev.* **D55** (1997) 1091, [hep-th/9608177](#).
- [32] K. Hashimoto, T. Hirayama and S. Moriyama, “*Symmetry Origin of Nonlinear Monopole*”, *JHEP* **0011** (2000) 014, [hep-th/0010026](#).
- [33] N. Ja. Vilenkin and A. U. Klimyk, “*Representation of Lie groups and special functions*”, Vol.1, Kluwer Academic Publishers, The Netherlands, 1991.